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MAPPINGS OF BAIRE SPACES INTO FUNCTION SPACES AND KADEČ RENORMING

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ABSTRACT
Assuming that there exists in the unit interval $[0,1]$ a coanalytic set of continuum cardinality without any perfect subset, we show the existence of a scattered compact Hausdorff space K with the following properties: (i) For each continuous map f on a Baire space B into $(C(K), \text{pointwise})$, the set of points of continuity of the map $f: B \to (C(K), \text{norm})$ is a dense G_δ subset of B, and (ii) $C(K)$ does not admit a Kadeč norm that is equivalent to the supremum norm. This answers the question of Deville, Godefroy and Haydon under the set theoretic assumption stated above.

1. Introduction

All topological spaces, except those in the Appendix, are Hausdorff in this note. For a compact space $K, C(K)$ denotes the Banach space of all real-valued continuous functions on K with the supremum norm. Besides the usual norm and weak topologies on $C(K)$, we also consider the topology τ_p of the pointwise convergence on K. After Debs [Deb], we say that a compact space K has the property $(\mathcal{N}^*)$ if, for each continuous map f from a Baire space B into $(C(K), \tau_p)$, the set of all points of continuity of the map $f: B \to (C(K), \text{norm})$ is a dense G_δ-subset of B. In his Paris thesis [Dev 1], Deville has shown that Eberlein compact spaces

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have property ($\mathcal{N}^*$), and this result has been generalized by Debs [Deb] to Corson compact spaces.

Recently, Deville and Godefroy [DG] have obtained a further generalization by suitable renorming of function spaces. We say that the norm $\|\cdot\|$ of a Banach space X is a **Kadeč norm** if, on the unit sphere $\{x \in X\colon \|x\| = 1\}$, the norm topology coincides with the weak topology. In case $X = C(K), (K \text{compact})$ a norm $\|\cdot\|$, which is equivalent to the supremum norm, is a **τ_p-Kadeč norm** if the norm topology (= the topology of uniform covergence) and τ_p coincide on the $\|\cdot\|$-unit sphere. Deville and Godefroy observed that an equivalent locally uniformly convex norm, which is τ_p lower semicontinuous, is a τ_p-Kadeč norm and that, if $C(K)$ admits an equivalent τ_p lower semicontinuous τ_p-Kadeč norm, then K has the property ($\mathcal{N}^*$). They have shown further that such renorming is indeed possible for a class of compact spaces K strictly larger than the class of Corson compact spaces. One of the problems posed by them (Problem 1) is the following: Does there exist a compact space K with the property ($\mathcal{N}^*$) such that no equivalent norm on $C(K)$ is locally uniformly convex and τ_p lower semicontinuous?

For scattered compact spaces K, Deville [Dev 2] has shown that if $K^{(\omega_1)} = \emptyset$ then K has property ($\mathcal{N}^*$), and a recent paper by Haydon and Rogers [HR] proves that the same condition implies the existence of an equivalent locally uniformly convex norm. We remark here that for a scattered compact space K, the weak topology and τ_p coincide on bounded subsets of $C(K)$. Hence each equivalent norm on $C(K)$ is τ_p lower semicontinuous and each equivalent Kadeč norm is a τ_p-Kadeč norm. In [H], Haydon has constructed a compact scattered space K such that $K^{(\omega_1)}$ is a singleton (hence $K^{(\omega_1+1)} = \emptyset$), K fails property ($\mathcal{N}^*$) and $C(K)$ has no equivalent strictly convex norm nor equivalent smooth norm. Pondering on this state of affairs, Haydon reiterates the question of Deville and Godefroy specifically for compact scattered spaces.

In [JNR 2], it is shown that if K is a compact space such that $(C(K), \tau_p)$ is σ-fragmentable (for definition, see Section 3), then K has property ($\mathcal{N}^*$). If $C(K)$ has a τ_p lower semicontinuous and τ_p-Kadeč renorming, then $(C(K), \tau_p)$ is σ-fragmentable [JNR 2]. It was even conjectured by one of us that, if K has property ($\mathcal{N}^*$), then $(C(K), \tau_p)$ is σ-fragmentable.

In this note we show that the above conjecture is false and the answer to the question of Deville and Godefroy and Haydon is affirmative provided we assume the following statement (independent of ZFC axioms):

(∗) There exists a coanalytic set in the unit interval $[0,1]$ of continuum cardinality without any perfect subset.

Statement (∗) implies the continuum hypothesis CH (use [J; Theorem 94, p.507 and Corollary 3, p.519]) and it is satisfied in Gödel's universe ($V = L$) [J; Corollary 2, p.529 and Theorem 3.4, p.110].

Our main result is the following.

1.1 THEOREM: *Assume (∗). Then there exists a compact scattered space K with property ($\mathcal{N}^*$) such that $C(K)$ has no Kadeč norm equivalent to the supremum norm. Furthermore $(C(K), \tau_p)$ is not σ-fragmentable.*

More specifically, the compact scattered space K of Theorem 1.1 is obtained from a theorem of Kunen (recorded in Section 3) applied to a set with property (∗). The fact that $C(K)$ does not admit any equivalent Kadeč norm (provided K is uncountable) follows immediately from Kunen's theorem and does not require property (∗). The bulk of the proof of Theorem 1.1 is in showing that K has property ($\mathcal{N}^*$). In Section 2, we prove a theorem on scattered spaces which may be of independent interest: in order to prove property ($\mathcal{N}^*$) of a scattered compact space K, one only needs to look at maps from Baire spaces to the space of $\{0,1\}$-valued continuous functions on K, i.e. the space of all open and closed subsets of K. In Sections 4 and 5 some auxiliary lemmas are established, and finally, in Section 6, the proof of Theorem 1.1 is completed by showing that property ($\mathcal{N}^*$) holds for the one-point compactification of a set with property (∗) endowed with the Kunen topology.

However, we also show, in Section 7, that, under a different set theoretic assumption, viz. the existence of a precipitous ideal in ω_1 (for definitions, see Section 7), the same procedure applied to an arbitrary uncountable subset of $[0,1]$ yields a compact scattered space without property ($\mathcal{N}^*$). In Section 8, we collect further comments on the subject of the present note. In the Appendix, we compare our definition of property ($\mathcal{N}^*$) to other possible definitions some of which have appeared in literature. In turns out that they are all equivalent. In particular, one can limit Baire spaces to completely regular ones.

This paper was written while the second author was visiting the University of Washington; he would like to express his gratitude to the Department of Mathematics for its hospitality.

2. Reduction to Zero-One Functions

In this section we show that property $(\mathcal{N}^*)$ for compact scattered spaces K can be reduced to a property concerning the space of all $\{0,1\}$-valued continuous functions on K. We say that a compact space K **fails property $(\mathcal{N}^*)$ with respect to a Baire space B** if there exists a continuous map $u: B \to (C(K), \tau_p)$ such that the set (necessarily a G_δ-set) of all points of continuity of the map $u: B \to (C(K), \text{norm})$ is not dense in B. Since B is Baire, this condition is equivalent to the existence of an $\varepsilon > 0$ and a non-empty open subset U of B such that diam $u(V) > \varepsilon$ for each non-empty open subset V of U.

2.1 LEMMA: *Let K be a compact scattered space. If K fails property $(\mathcal{N}^*)$ with respect to a Baire space B, then there exists a non-empty open subset E of B and a continuous map $v: E \to (C(K), \tau_p)$ such that $v(b)(K) \subset \{0,1\}$ for each $b \in E$ and v is not constant on each non-empty open subset of E.*

Proof: As seen above, the hypothesis implies the existence of a continuous map $u: B \to (C(K), \tau_p)$ and a non-empty open subset U of B such that diam $u(V) > \varepsilon$ for each non-empty open subset V of U. Since U is Baire, by shrinking U if necessary, we may assume that $u(U)$ is bounded, i.e. for some $c > 0$, $u(b)(K) \subset (-c, c)$ for each $b \in U$. Let $J = (-c, c)$.

Let $\mathcal{F}$ be the countable collection of all finite sets F of rational numbers in J such that the distance of each point of J to F is less than $\varepsilon/2$. For each $F \in \mathcal{F}$ and $n \in \mathbb{N}$, the set

$$A_n(F) = \{b \in B\colon \text{dist}(u(b)(K), F) \geq 1/n\}$$

is closed. Furthermore the countable family

$$\{A_n(F)\colon F \in \mathcal{F} \text{ and } n \in \mathbb{N}\}$$

covers B. For, if $b \in B$, then since $u(b)(K)$ is scattered it is a countable compact subset of J. Hence there is an $F \in \mathcal{F}$ with $u(b)(K) \cap F = \emptyset$. It follows that $b \in A_n(F)$ for some $n \in \mathbb{N}$. Since U is a Baire space, there is a non-empty open set G such that $G \subset A_n(F) \cap U$ for some $F \in \mathcal{F}$ and $n \in \mathbb{N}$. Let $F = \{q_1, \ldots, q_m\}$ where

$$-c = q_0 < q_1 < \cdots < q_m < q_{m+1} = c,$$

and let $\varphi: J \backslash F \to F^* = F \cup \{c\}$ be the continuous map given by $\varphi(x) = q_i$ whenever $x \in (q_{i-1}, q_1), i = 1, \ldots, m+1$. Then $b \mapsto \varphi \circ u(b)$ defines a continuous function $w: G \to (C(K), \tau_p)$. Notice that if $a, b \in G$ and $w(a) = w(b)$, then for each $x \in K$, $u(a)(x)$ and $u(b)(x)$ are both in (q_{i-1}, q_i) for some $i, i = 1, \dots, m+1$, and therefore $|(a)(x) - u(b)(x)| < \varepsilon$. It follows that $\|u(a) - u(b)\| < \varepsilon$ whenever $w(a) = w(b)$, and consequently w is not constant on each non-empty open subset V of G since diam $u(V) > \varepsilon$.

To conclude the proof, place the discrete space F^* in $\{0,1\}^k$ for some $k \in \mathbb{N}$, and let p_i: $\{0,1\}^k \to \{0,1\}$ be the i-th projection. If we denote by v_i the map $b \mapsto p_i \circ w(b)$ for $b \in G$, then there is at least one i for which there exists a non-empty open subset E of G with the property that v_i is not constant on each non-empty open subset of E. For, otherwise, there would be a decreasing sequence $G_1 \supset G_2 \supset \cdots \supset G_k$ of non-empty open subsets of G such that v_i is constant on G_i for each i, and consequently w would be constant on G_k. This concludes the proof. ∎

From Lemma 2.1, we immediately obtain the following characterization of property $(\mathcal{N}^*)$ for scattered compact spaces. We denote by $C(K, \{0,1\})$ the space of all $\{0,1\}$-valued continuous functions on K.

2.2 THEOREM*: *A compact scattered space K has property $(\mathcal{N}^*)$ if and only if each continuous map of a Baire space B into $(C(K, \{0,1\}), \tau_p)$ is constant on some non-empty open subset of B.*

2.3 Remark: Let K be as in Lemma 2.1, and fix an arbitrary point p in K. Then we can make the map v in Lemma 2.1 satisfy an additional condition: $v(b)(p) = 0$ for each $b \in E$. Indeed, one of the sets $\{b \in E: v(b)(p) = d\}$, $d = 0, 1$, has non-empty interior. ∎

3. A Theorem of Kunen

Our example of Theorem 1.1 is based on the following theorem due to K. Kunen; a proof can be found in [Ne].

3.1 THEOREM: *Assume CH and let $X \subset [0,1]$. Then there exists a locally compact, locally countable topology $\mathcal{T}$ on X, stronger than the Euclidean topology, such that, if K is the one-point compactification of $(X, \mathcal{T})$, then the function space $C(K)$ is hereditarily Lindelöf in the weak topology.*

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* After the present note was submitted, we learned that R. Haydon had also proved this theorem in his Choquet Seminar talk, December 1989.

Let $(X, \mathcal{T})$ be a space as in Theorem 3.1. Then the local countability implies that $(X, \mathcal{T})$ is scattered and therefore its one-point compactification K is also scattered. If the Banach space $C(K)$ admits an equivalent Kadeč norm $|||\cdot|||$, then the $|||\cdot|||$-unit sphere is norm separable since it is weakly (and hence norm) Lindelöf. Thus $C(K)$ is separable or equivalently K is metrizable. Since K is also scattered K is countable. Therefore if X in Theorem 3.1 is uncountable, then $C(K)$ has no Kadeč norm that is equivalent to the supremum norm.

One can state a slightly stronger conclusion than above using the notion of σ-fragmentability introduced in [JNR 1]. A topological space (T, τ) is said to be **σ-fragmented** by a metric ρ on T, if for each $\varepsilon > 0$, T can be written as $T = \bigcup\{T_n: n \in \mathbb{N}\}$ where each T_n has the property: each non-empty subset of T_n contains a non-empty relatively τ-open subset of ρ-diameter less than ε. We shall simply say that a Banach space E is **σ-fragmentable** if E with the weak topology is σ-fragmented by the norm metric. It is known that, if E admits an equivalent Kadeč norm, or more generally, if E as a subset of E^{**} is obtained from a family of weak* Borel subsets of E^{**} by the Souslin operation (i.e. operation $\mathcal{A}$), then E is σ-fragmentable [JNR 1].

3.2 LEMMA: *If a Banach space E is σ-fragmentable and hereditarily Lindelöf in its weak topology, then E is separable.*

Proof: Assume that E is not separable. Then for some $\varepsilon > 0$ there exists an uncountable subset D of E such that $\|x - y\| \geq \varepsilon$ whenever $x, y \in D$ and $x \neq y$. Since E is σ-fragmentable, $D = \bigcup\{D_n: n \in \mathbb{N}\}$, where, for each n, each non-empty subset of D_n contains a relatively weak open subset of diameter less than ε, i.e. each non-empty subset of D_n contains a weakly isolated point. Since D is uncountable, D_n is uncountable for some n. Let A be the set of points of condensation of D_n in E. Then since E is hereditarily Lindelöf in the weak topology, $D_n \setminus A$ is countable and therefore $D_n \cap A$ is non-empty with no weakly isolated point. This contradiction proves the lemma. ∎

Taking into account the remark directly after Theorem 3.1, we have the following corollary.

3.3 COROLLARY: *In Theorem 3.1, if X is uncountable, then the Banach space $C(K)$ is not σ-fragmentable.*

4. A Lemma on Families of Countable Compacta

We shall denote by $\mathcal{C}(I)$ the space of compact countable subsets of the unit interval I with the topology induced by the Hausdorff metric (cf. [Ku; §42]). Recall that a standard countable base for $\mathcal{C}(I)$ consists of subsets of $\mathcal{C}(I)$ of the form

$$\langle J_1, \dots, J_n \rangle = \{L \in \mathcal{C}(I) \colon L \subset J_1 \cup \cdots \cup J_n \text{ and } L \cap J_i \neq \emptyset \text{ for } i = 1, \dots, n\}$$

where $J_i, i = 1, \dots, n$, are open intervals with rational end points.

4.1 LEMMA: *Let S be a Baire space and let $N \colon S \to \mathcal{C}(I)$ be a continuous map such that $N(s) \setminus N(t)$ is compact for all $s, t \in S$ and, for each non-empty open subset V of S,*

$$(1) \qquad N(V) = \bigcup \{N(s) \colon s \in V\} \text{ is uncountable.}$$

Then there exist a dense G_δ-subset T of S and a continuous map $f \colon T \to I$, that is not constant on each non-empty open subset of T, such that $f(t) \in N(t)$ for each $t \in T$.

Proof: For each non-empty open set V in S, let $D(V)$ denote the set of all condensation points of $N(V)$ in I (see(1)), and for $s \in S$, let

$$D(s) = \bigcap \{D(V) \colon V \text{ is a neighborhood of } s\}.$$

By (1), $D(V) \neq \emptyset$ for each open subset V of S, and

$$\{D(V) \colon V \text{ is an open neighborhood of } s\}$$

is a downward filtered family of compact sets in I. Hence $D(s) \neq \emptyset$. To see that $D(s) \subset N(s)$, let G be a neighborhood of $N(s)$ in I. Then by continuity of N, there is a neighborhood V of s with $N(V) \subset G$ whence $D(s) \subset D(V) \subset \overline{G}$. Since G is arbitrary, we now have

$$(2) \qquad \emptyset \neq D(s) \subset N(s) \qquad \text{for all } s \in S.$$

Let

$$(3) \qquad f(s) = \min D(s) \qquad \text{for } s \in S.$$

Then the map $f: S \to I$ is lower semicontinuous. For, if $a < f(s)$, then $D(s) \subset (a,1]$ and therefore $D(V) \subset (a,1]$ for some neighborhood V of s. It follows that $f(t) \in D(t) \subset D(V) \subset (a,1]$ for each $t \in V$. By a standard theorem (cf. [E] 1.7.14(b)), the set T of all points of continuity of f is a dense G_δ-subset of S. Of course, then $f|T$ is continuous, and $f(t) \in N(t)$ for each $t \in T$.

It remains to show that for each non-empty open set U in S, f takes at least two values on $U \cap T$. Let $t \in U \cap T$. For each $i \in \mathbb{N}$, let I_i be the open $1/i$-interval around $f(t)$, and put

$$F_i = \{s \in S: N(s) \cap I_i \subset N(t)\}.$$

Then the set F_i is closed, because

$$S \setminus F_i = \{s \in S: N(s) \cap (I_i \setminus N(t)) \neq \emptyset\}$$

is open by the continuity of N. Furthermore, if $s \in S$, then, since $N(s) \setminus N(t)$ is a compact set not containing $f(t)$, $I_i \cap (N(s) \setminus N(t)) = \emptyset$ for some i, i.e. $s \in F_i$. By hypothesis $S = \bigcup\{F_i: i \in \mathbb{N}\}$ is a Baire space. Hence there are a non-empty open set V in S and an $i \in \mathbb{N}$ such that $V \subset F_i \cap U$. Then since $N(V) \cap I_i \subset N(t)$ and the set $N(t)$ is countable, each point of condensation of $N(V)$ lies off the interval I_i, i.e. $D(V) \subset I \setminus I_i$. Consequently, if $s \in V \cap T$, then $f(s) \in D(s) \subset D(V) \subset I \setminus I_i$ and so $f(s) \neq f(t) \in I_i$. This completes the proof. ∎

5. A Remark on Sets with Property (*)

We shall need the following lemma, which is a slight strengthening of the well-known fact that sets with property (*) are meager in a very strong sense.

5.1 Lemma: *Let $X \subset I = [0,1]$ be a set with property (*). Then each continuous map $f: T \to X$ defined on a completely regular Baire space T is constant on some non-empty open set in T.*

Proof: We prove by contradiction. Assume that a map $f: T \to X$ is not constant on each non-empty set in T. Then, following a standard argument, we show that property (*) for X is violated.

Let $\overline{f}: \beta T \to I$ be the extension of f to the Stone-Čech compactification βT of T, and let $M = \overline{f}^{-1}(X)$. Then since $I \setminus X$, being analytic, is a Souslin-$\mathcal{F}$ set in I, its preimage $\beta T \setminus M$ under $\overline{f}$ is a Souslin-$\mathcal{F}$ set in βT. (Here and elsewhere, we use the results and terminology of [RJ].)

It follows that $\beta T\backslash M$ (and therefore M) has the Baire property, i.e. $M = (U\backslash N_1) \cup N_2$, where U is open in βT and N_1, N_2 are first category sets in βT. Furthermore, since T is a Baire space and $T \subset M \subset \beta T$, M is dense in βT and is a Baire space. It is easy to verify that N_2 is of first category in M, and therefore $U\backslash N_1$ is dense in M and also in βT. Consequently $U\backslash N_1$ contains a dense G_δ-subset G of βT, and $\overline{f}$ is not constant on each non-empty open set in G because $T \cap G$ is dense in T.

We may now shift our attention to G, and for simplicity, the map $\overline{f}$: $\beta T \to X$ will be denoted by f. Let $\{G_n: n \in \mathbb{N}\}$ be a sequence of open sets in βT such that $G = \bigcap\{G_n: n \in \mathbb{N}\}$. Let D be the set of all finite sequences of 0's and 1's. By induction on the length of members of D, we choose a family $\{H_d: d \in D\}$ of open sets in βT such that

(i) $H_{d0} \cup H_{d1} \subset H_d$ for each $d \in D$;

(ii) $f(\overline{H}_{d0}) \cap f(\overline{H}_{d1}) = \emptyset$ for each $d \in D$;

(iii) If the length of d is n, then $\overline{H}_d \subset G_n$.

To get the induction started, we let $H_\emptyset = G_0 = \beta T$, where $\emptyset$ denotes the empty sequence of length 0. Suppose we have already chosen H_d for all $d \in D$ of length $\leq n$. Fix a $d \in D$ of length n. Then there are two points m_0 and m_1 in $H_d \cap G$ so that $f(m_0) \neq f(m_1)$. Then we can choose open neighborhoods H_{di} of m_i $(i = 0, 1)$ so that (i) and (ii) as well as $\overline{H}_{di} \subset G_{n+1}$ $(i = 0, 1)$ are satisfied. Now for each $n \in \mathbb{N}$, let

$$Z_n = \bigcup\{\overline{H}_d: d \in D \text{ with length } n\} \subset G_n$$

and let

$$Z = \bigcap\{Z_n: n \in \mathbb{N}\} \subset \bigcap\{G_n: n \in \mathbb{N}\} = G \subset M.$$

Clearly Z is compact and hence $f(Z)$ is a compact subset of X. To see that the cardinality of $f(Z)$ is continuum, it is sufficient to observe that if $u, v \in \{0,1\}^{\mathbb{N}}$ and $u \neq v$, then by (ii),

$$f(\bigcap\{\overline{H}_{u|n}: n \in \mathbb{N}\}) \cap f(\bigcap\{\overline{H}_{v|n}: n \in \mathbb{N}\}) = \emptyset.$$

(Here if $u = (i_1, i_2, \ldots)$ then $u|n = (i_1, \ldots, i_n)$.) This contradicts property (*), and the proof is complete. ∎

6. Proof of Theorem 1.1

Before we proceed to the proof of Theorem 1.1, we need one more lemma.

6.1 LEMMA: *If $\mathcal{A}$ is a family of compact open subsets of a topological space such that $\bigcup \mathcal{A}$ is countable, then $\mathcal{A}$ is countable.*

Proof: Suppose that $\mathcal{A}$ is uncountable. Since $\bigcup \mathcal{A}$ is countable, there is a countable subfamily $\mathcal{A}_0$ of $\mathcal{A}$ such that $\bigcup \mathcal{A}_0 = \bigcup \mathcal{A}$. Then each member A of $\mathcal{A}$ is covered by a finite subfamily of $\mathcal{A}_0$. Therefore there exist an uncountable subfamily $\mathcal{B}$ of $\mathcal{A}$ and a finite sequence $\{A_i\colon i = 1, \dots, n\}$ in $\mathcal{A}_0$ such that each member of $\mathcal{B}$ is contained in $L = \bigcup\{A_i\colon i = 1, \dots, n\}$. Being compact and countable, L is a compact metrizable space and hence $C(L)$ is separable. However, $\{\chi_B\colon B \in \mathcal{B}\}$ is an uncountable subset of $C(L)$ with the distance between any distinct members at least one. This contradiction proves the lemma. ∎

Proof of Theorem 1.1: Let X be a subset of the unit interval I with property (*), and let K be the one point compactification, with p_∞ the point at infinity, of the space $(X, \mathcal{T})$ with Kunen's topology as described in Theorem 3.1. Recall that the space K is scattered and $C(K)$ has no Kadeč norm equivalent to the supremum norm. In fact, $C(K)$ is not even σ-fragmentable (Corollary 3.3). It remains to verify property $(\mathcal{N}^*)$ for K.

Suppose that K does not have property $(\mathcal{N}^*)$. Then as explained in Section 1 (and proved in the Appendix), K fails $(\mathcal{N}^*)$ with respect to a completely regular Baire space B (cf. Section 2). By Lemma 2.1 and Remark 2.3, there exist a non-empty open set E in B and a continuous map $v\colon E \to (C(K), \tau_p)$ such that, for each $t \in E$, $v(t)(K) \subset \{0, 1\}$, $v(t)(p_\infty) = 0$ and v is not constant on each non-empty open set in E. For each $t \in E$, let

$$N(t) = \{x \in K\colon v(t)(x) = 1\} \subset X \subset I.$$

Then $N(t)$ is a compact open subset of $(X, \mathcal{T})$, and it is countable because the topology $\mathcal{T}$ is locally countable. Furthermore, if $t, s \in E$, then $N(s)\backslash N(t)$ is $\mathcal{T}$-compact, and, since $\mathcal{T}$ is stronger than the usual Euclidean topology of X, both $N(t)$ and $N(s)\backslash N(t)$ are compact subsets of I in the usual topology, i.e. $N(t), N(s)\backslash N(t) \in \mathcal{C}(I)$ for all $s, t \in E$ (cf. Section 4).

Now for each point $x \in I$, the set $\{t \in E\colon x \in N(t)\}$ is open by the continuity of v. Hence for each subset A of I, $\{t \in E\colon N(t) \cap A \neq \emptyset\}$ is open and $\{t \in E\colon N(t) \subset A\}$ closed in E. Therefore the map $t \mapsto N(t)$ is Borel-measurable where $\mathcal{C}(I)$ is given the topology of the Hausdorff metric (see Section 4). Since E, being open in a Baire space B, is Baire, there exists a dense G_δ-subset S of E such that the restriction of N to S is continuous [Ku; §32 II]. Let V be a non-empty open subset of E, then $v(V \cap S)$ is uncountable. For, if $v(V \cap S)$ were countable, then, since S is a Baire space, there is a non-empty open subset W of E such that v is constant on $W \cap S$ and hence on W by the continuity of v. This contradicts our choice of E and v. Hence $v(V \cap S)$ is uncountable, and since, for each $s \in V \cap S$, $N(s)$ is a compact open subset of $(X, \mathcal{T})$, we conclude from Lemma 6.1 that $N(V \cap S) = \bigcup\{N(s): s \in V \cap S\}$ is uncountable.

We may now apply Lemma 4.1 to our map $N: S \to \mathcal{C}(I)$ to obtain a dense G_δ subset T of S and a continuous map $f: T \to X \subset I$ which is not constant on each non-empty open subset of T. This contradicts the conclusion of Lemma 5.1 since X has property (*), and the proof is complete. ∎

7. Precipitous Ideals and the Kunen Construction

In this section we show that under a set theoretic hypothesis different from (*), the compact space K obtained by applying Theorem 3.1 (the Kunen construction) to an uncountable subset X of $[0,1]$ fails the property $(\mathcal{N}^*)$. This is directly opposite to the conclusion we reached in the last section under (*).

Let S be a set. A family $\mathcal{I}$ of subsets of S is called an **ideal** if $A, B \in \mathcal{I}$ implies $C \in \mathcal{I}$ for each $C \subset A \cup B$. The ideal $\mathcal{I}$ is called a **σ-ideal** if $\mathcal{I}$ is closed under countable unions, and $\mathcal{I}$ is called **non-principal** if it contains all finite subsets of S. We denote by $\mathcal{I}^+$ the family of all subsets A of S not belonging to $\mathcal{I}$. (Informally the members of $\mathcal{I}$ are "small" and the members of $\mathcal{I}^+$ are "large"). To an ideal $\mathcal{I}$, one can associate a two-person game $G(\mathcal{I})$ (due to Galvin; see [BTW]). The players "Void" and "Non-void" alternatively pick sets $A_1, A_2, \ldots$ in $\mathcal{I}^+$ such that $A_i \supset A_{i+1}$ for all i. "Void" wins the game if $\bigcap\{A_i: i \in \mathbb{N}\} = \emptyset$; otherwise "Non-void" wins. A non-principal ideal $\mathcal{I}$ is called **precipitous** if "Void" does not possess a winning strategy for the game $G(\mathcal{I})$. It is easy to check that each precipitous ideal is a σ-ideal.

An ideal $\mathcal{I}$ of subsets of a topological space T is said to be **local** if, for a subset A of T, $A \in \mathcal{I}$ provided that A is *locally* in $\mathcal{I}$, i.e. for each $t \in A$ there is a neighborhood U of t such that $A \cap U \in \mathcal{I}$.

7.1 LEMMA: *Let T be a topological space. If there is a precipitous ideal $\mathcal{I}$ of subsets of T, which is local, then there exist a Baire space B and a continuous* *map $f\colon B \to T$ which is not constant on each non-empty open set in T.*

Proof: The proof follows the idea of Krom and Frankiewicz–Kunen [Kr] [FK]. Given $A \subset T$, let $V = \bigcup\{W\colon W \text{ open in } T \text{ and } W \cap A \in \mathcal{I}\}$. Then $A^* = A \setminus V$ is the largest relatively closed subset of A such that

(1) $\qquad U \cap A^* \in \mathcal{I}^+ \qquad$ whenever U is open and $U \cap A^* \neq \emptyset$

If $A^* = \emptyset$, then A is locally in $\mathcal{I}$ and so by hypothesis $A \in \mathcal{I}$. The converse is clear. We note that $(A^*)^* = A^*$ and $A^* \subset B^*$ whenever $A \subset B$. Let M be the collection of all sequences $\gamma = \langle E_1, E_2, \ldots \rangle$ in $\mathcal{I}^+$ such that

(2) $\qquad E_i \supset E_{i+1}$ for all $i \in \mathbb{N}$ and $E_i = E_i^*$ for infinitely many i's.

As in Krom [Kr], we give M the metrizable topology induced by the product topology of $(\mathcal{I}^+)^{\mathbb{N}}$, where $\mathcal{I}^+$ is given the discrete topology. The standard base for this topology consists of all the sets of the form

$$N(E_1, \ldots, E_n) = \{\gamma \in M\colon \gamma \text{ extends } \langle E_1, \ldots, E_n \rangle\},$$

where $E_1 \supset E_2 \supset \cdots \supset E_n$ and $E_i \in \mathcal{I}^+$ for $i = 1, \ldots, n$. Let B be the subspace of the product $T \times M$ defined by

$$B = \{(x, \gamma)\colon \gamma = \langle E_1, E_2, \ldots \rangle \in M \text{ and } x \in \bigcap_{i=1}^{\infty} E_i\}$$

and let $f\colon N \to T$ be the restriction of the projection: $T \times M \to T$. Then we show that

(a) B is a Baire space, and

(b) f is not constant on each non-empty open set in B.

First we observe that, for each open set V in T and a finite decreasing sequence $\langle E_1, \ldots, E_n \rangle$ in $\mathcal{I}^+$,

(3) $\qquad B \cap (V \times N(E_1, \ldots, E_n)) \neq \emptyset \qquad$ if and only if $V \cap E_n \in \mathcal{I}^+$.

Indeed, if $(x, \gamma) \in B \cap (V \times N(E_1, \ldots, E_n))$, then (by (2)), for some $k > n$, the k-th term E_k of γ satisfies $x \in V \cap E_k^* = V \cap E_k \subset V \cap E_n$. Since $V \cap E_k^* \in \mathcal{I}^+$ by (1), $V \cap E_n \in \mathcal{I}^+$. Conversely suppose $V \cap E_n \in \mathcal{I}^+$ and let $A = (V \cap E_n)^*$.

Then $\gamma = \langle E_1, \dots, E_n, A, A, \dots\rangle \in M$ and if $x \in A \subset V$, then $(x, \gamma) \in B \cap (V \times N(E_1, \dots, E_n))$.

In order to show (a), assume that $\{G_i \colon i \in \mathbb{N}\}$ is a sequence of dense open subsets of B, V an open set in T and that $\langle E_1, \dots, E_n\rangle$ is a finite decreasing sequence in $\mathcal{I}^+$ such that $B \cap (V \times N(E_1, \dots, E_n)) \neq \emptyset$ or $V \cap E_n \in \mathcal{I}^+$. We must show that $(V \times N(E_1, \dots, E_n)) \cap \bigcap\{G_i \colon i \in \mathbb{N}\} \neq \emptyset$. Consider the following strategy of "Void" for the game $G(\mathcal{I})$. "Void" begins by $A_0 = (V \cap E_n)^*$. Suppose "Non-void" plays $A_1 \in \mathcal{I}^+$ with $A_1 \subset A_0$. Then $B \cap (V \times N(E_1, \dots, E_n, A_1)) \neq \emptyset$ by (3). Since G_1 is dense and open in B, one may choose an open set $V_1 \subset V$ and a decreasing finite sequence $\langle A_2^1, \dots, A_2^{n_1}\rangle$ in $\mathcal{I}^+$ such that $A_1 \supset A_2^1$ and

$$\emptyset \neq B \cap (V_1 \times N(E_1, \dots, E_n, A_1, A_2^1, \dots, A_2^{n_1})) \subset G_1$$

After this preliminary consideration "Void" plays $A_2 = (V_1 \cap A_2^{n_1})^*$. Then the entire process is repeated. In general, suppose that "Non-void" plays $A_{2i+1} \in \mathcal{I}^+$ so that $A_{2i+1} \subset A_{2i} = (V_i \cap A_{2i}^{n_i})^*$. Then by (3),

$$B \cap (V_i \times N(E_1, \dots, E_n, A_1, A_2^1, \dots, A_{2i}^{n_i}, A_{2i+1})) \neq \emptyset$$

and since G_{i+1} is dense and open in B, there are an open set V_{i+1} and a finite decreasing sequence $\langle A_{2i+2}^1, \dots, A_{2i+2}^{n_i+1}\rangle$ in $\mathcal{I}^+$ such that $A_{2i+1} \supset A_{2i+2}^1$ and

$$(4)\quad \emptyset \neq B \cap (V_{i+1} \times N(E_1, \dots, E_n, A_1, \dots, A_{2i+1}, A_{2i+2}^1, \dots, A_{2i+2}^{n_i+1})) \subset G_{i+1}.$$

"Void" then plays $A_{2i+2} = (V_{i+1} \cap A_{2i+2}^{n_i+1})^*$. Since this strategy of "Void" cannot be a winning one, there is a sequence $\{A_{2i+1}\}$ of moves by "Non-void" that fails the scheme of "Void", i.e. $\bigcap\{A_{2i+1} \colon i \in \mathbb{N}\} \neq \emptyset$. If $x \in \bigcap\{A_{2i+1} \colon i \in \mathbb{N}\}$ and $\gamma = \langle E_1, \dots, E_n, A_1, \dots, A_{2i}^{n_i}, A_{2i+1}, A_{2i+2}^1, \dots\rangle$, then $x \in \bigcap\{V_i \colon i \in \mathbb{N}\}$ and $(x, \gamma) \in B$. Consequently, by (4), $(x, \gamma) \in (V \times N(E_1, \dots, E_n)) \cap \bigcap\{G_i \colon i \in \mathbb{N}\}$, and (a) is proved.

To see (b), let $U = B \cap (V \times N(E_1, \dots, E_n)) \neq \emptyset$ be an open set in B as in (3). Since $V \cap E_n \in \mathcal{I}^+, V \cap E_n^* \neq \emptyset$ or equivalently $V \cap E_n^* \in \mathcal{I}^+$ (by(1)). Since $\mathcal{I}$ is non-principal, there are two distinct points t_1 and t_2 in $V \cap E_n^*$. Let V_1 and V_2 be open neighborhoods of t_1 and t_2 respectively such that $V_1 \cap V_2 \neq \emptyset$ and $V_j \subset V$ $(j = 1, 2)$. Then for each j, $V_j \cap E_n^* \in \mathcal{I}^+$ (by(1)), and hence $\emptyset \neq U_i = B \cap (V_j \times N(E_1, \dots, E_n)) \subset U$. Since $f(U_j) \subset V_j$, f is not constant on U. ∎

For the rest of this section we make the following assumption:

(**) There is a precipitous ideal of subset of ω_1.

The statement (**) is equiconsistent with the existence of a measurable cardinal (see [J; Theorem 86, p. 447]).

7.2 THEOREM: *Assume* (**). *If K is a compact space with property* ($\mathcal{N}^*$), *then each subset of $C(K)$ that is hereditarily Lindelöf in τ_p (the pointwise topology) is norm-separable.*

Proof: Let H be a subset of $C(K)$ which is hereditarily Lindelöf in τ_p. If H is not norm-separable, then there exists an $\varepsilon > 0$ and an uncountable subset $T \subset H$ such that $\|t - t'\| \geq \varepsilon$ whenever $t, t' \in \dot{T}$ and $t \neq t'$. We may assume that $|T| = |\omega_1|$. Then by (**), there is a precipitous ideal $\mathcal{I}$ of subsets of T. If $A \subset T$ and A is locally in $\mathcal{I}$, relative to τ_p, then by hypothesis A is covered by a countable subfamily of $\{V\colon V$ is τ_p-open and $V \cap A \in \mathcal{I}\}$. Since, as remarked, $\mathcal{I}$ is a σ-ideal, $A \in \mathcal{I}$.

It follows from Lemma 7.1 that there is a Baire space B and a continuous map $f\colon B \to (T, \tau_p)$ which is not constant on each non-empty open set in B. Since K has property ($\mathcal{N}^*$), $f\colon B \to (T$, norm) is continuous at some $b \in B$. Therefore, there is an open neighborhood U of b with diam $f(U) < \varepsilon$. By the choice of the set T, $f(U)$ is a singleton. This contradiction proves the theorem. ∎

7.3 COROLLARY: *Assume* (**). *In Theorem 3.1, if $X \subset [0,1]$ is uncountable, then K fails property* ($\mathcal{N}^*$).

Proof: By Theorem 3.1, $(C(K), \tau_p)$ is hereditarily Lindelöf. Hence, if K has property ($\mathcal{N}^*$), then $C(K)$ is norm-separable by Theorem 7.2, which implies that K is metrizable and therefore countable. ∎

8. Comments

In the proof of Theorem 1.1 in Section 6, property ($*$) enters only through Lemma 5.1. If there is a subset X of $[0,1]$ for which the conclusion of Lemma 5.1 holds for certain Baire spaces, then the corresponding space K obtained by applying Theorem 3.1 (using only CH) should have property ($\mathcal{N}^*$) with respect to some class of Baire spaces. We illustrate this principle with two examples.

Example 1: Let us identify the set P of all irrational numbers in $[0,1]$ with $\mathbb{N}^{\mathbb{N}}$. We define a partial ordering $\prec$ in P as follows: $\alpha \prec \beta$ if $\alpha(n) < \beta(n)$ for all but a finite number of $n \in \mathbb{N}$ (see e.g. [Ku; §40 III]). Let X be a subset of P well-ordered by $\prec$ and of order type ω_1. If T is a space (necessarily Baire) such that $T \times T$ is a Baire space, then each continuous map $f\colon T \to X$ is constant on some non-empty open subset of T. To see this, assume the contrary. Then

$$G = \bigcap_n \bigcup_k \{(s,t) \in T \times T\colon f(s)(n+k) < f(t)(n+k)\}$$

is dense in $T \times T$. For, let U and V be non-empty open subsets of T and $s \in U$. Then since $\{x \in X\colon x \preceq f(s)\}$ is countable and since $f^{-1}(x)$ is nowhere dense by the assumption,

$$V \backslash \{r \in T\colon f(r) \preceq f(s)\} = V \cap \{t \in T\colon f(s) \prec f(t)\} \neq \emptyset.$$

If t is in this intersection, then $(s,t) \in (U \times V) \cap G$. This shows that G, which clearly is a G_δ-set, is dense in $T \times T$. The same is true of $G^{-1} = \{(s,t)\colon (t,s) \in G\}$. Since $T \times T$ is a Baire space, $G \cap G^{-1} \neq \emptyset$. But if $(s,t) \in G \cap G^{-1}$, then $f(s)$ and $f(t)$ are incomparable by $\prec$. This contradiction proves the claim. ∎

Let K be the compact scattered space obtained from the set X above by applying Theorem 3.1. Then K has property $(\mathcal{N}^*)$ with respect to spaces B with $B \times B$ Baire. The proof is just a repetition of the proof of Theorem 1.1. It is sufficient to note that, if E is an open set in B and T a dense G_δ-subset of E, then $T \times T$ is a Baire space provided $B \times B$ is Baire.

Example 2: Let C be a coanalytic set in $[0,1]$ which is not analytic, and let $\{C_\alpha\colon \alpha < \omega_1\}$ be the constituents of the set C [Ku; §39 VIII]. Since C is not analytic $\{\alpha\colon C_\alpha \neq \emptyset\}$ is a cofinal subset of ω_1 [Ku; §39 VIII Corollary 5a]. Let X be a set composed of one point from each non-empty C_α. Then for each analytic subset A of C, $A \cap X$ is countable by the covering theorem [Ku; §39 VIII Theorem 5]. We show that each continuous map $f\colon T \to X$ defined on a completely regular Baire space with the countable chain condition (CCC) is constant on some non-empty open subset of T. To prove this, we proceed as in the proof of Lemma 5.1. Let $\overline{f}\colon \beta T \to [0,1]$ be the extension of f and let $M = \overline{f}^{-1}(C)$. Then as before, M contains a set G which is dense G_δ in βT. Since βT has the CCC, each dense open set in βT contains a dense open σ-compact set. (Consider a maximal disjoint family of cozero sets in the open set.) Hence G contains a dense G_δ-set H of the form

$$H = \bigcap_m \bigcup_n K_{m,n},$$

where $K_{m,n}$ is compact for all $m, n \in \mathbb{N}$. It follows that $\overline{f}(H)$ is an analytic subset of C and therefore $f(H \cap T) \subset \overline{f}(H) \cap X$ is countable by the property of X. Since $H \cap T$, which is dense and G_δ in T, is a Baire space, for some $x \in f(H \cap T)$, $f^{-1}(x)$ has non-empty interior relative to $H \cap T$ and hence relative to T. ∎

Now apply Theorem 3.1 to the set $X \subset [0,1]$ of this example. Then the resulting compact scattered space K has property $(\mathcal{N}^*)$ with respect to Baire spaces with CCC. As in Example 1, the above is established by repeating the proof of Theorem 1.1 while making sure that the CCC is preserved at each stage. This entails, for instance in Theorem A.2 (Appendix), replacing "Baire space" with "Baire space with the CCC". (The CCC version of Theorem A.2 is true because in the proof the space E is dense in the Stone space $S(\mathcal{B})$.)

Appendix

Throughout this Appendix, topological spaces are not necessarily Hausdorff. We say that $\mathcal{N}(B, K; M)$ holds for topological spaces B, K and M if, whenever $f: B \times K \to M$ is separately continuous, there exists a dense G_δ-set D in B such that the map f is jointly continuous at each point of $D \times K$. It is easy to check that (cf. [Na]) a compact Hausdorff space K has property $(\mathcal{N}^*)$ in the sense of Section 1 if and only if $\mathcal{N}(B, K; \mathbb{R})$ holds for each Hausdorff Baire space B. However, other authors (e.g. [B]) define property $(\mathcal{N}^*)$ for a compact space K by demanding $\mathcal{N}(B, K; M)$ to hold for each Baire space B and each metrizable space M. We show here that the last condition is equivalent not only to our $(\mathcal{N}^*)$ but also to a less restricted form of $(\mathcal{N}^*)$, viz. $\mathcal{N}(E, K; \mathbb{R})$ for each Tychonoff (=completely regular T_1) Baire space E.

A.1 THEOREM: *Let K be a compact space and let B be a Baire space. If $\mathcal{N}(W, K; \mathbb{R})$ holds for each open subset W of B, then $\mathcal{N}(B, K; M)$ holds for each metrizable space M.*

Proof: Let $\{e_s: s \in S\}$ be an orthonormal system in a Hilbert space, and let

$$J(S) = \{re_s: r \in [0,1], s \in S\}.$$

Then the space $J(S)$ with the norm topology of the Hilbert space is the standard "hedgehog space" with $|S|$ "spines". Since each metrizable space is embedded in a countable product $J(S)^\infty$ for a suitable S, it is sufficient to show that, under the hypothesis of the theorem, $\mathcal{N}(B,L;J(S))$ holds for an arbitrary index set S.

Let $f\colon B\times K\to J(S)$ be separately continuous. As in [Na], it is enough to show that, for each ε, $0<\varepsilon<1$, and for each non-empty open subset U of B, there exists $u\in U$ such that each $(u,k)\in\{u\}\times K$ has a neighborhood H in $B\times K$ such that diam $f(H)\le 3\varepsilon$. For each $b\in B$, the set

$$S(b)=\{s\in S\colon f(\{b\}\times K)\cap\{re_s\colon r>\varepsilon\}\neq\emptyset\}$$

is finite by the compactness of K. If $F_n=\{b\in B\colon |S(b)|\le n\}$, then $B=\bigcup\{F_n\colon n=0,1,2,\ldots\}$. Since B is Baire and each F_n closed, there are non-negative integer n and a non-empty open set V in B such that $V\subset U\cap F_n\backslash F_{n-1}$ (let $F_{-1}=\emptyset$). If $n=0$, then diam $f(V\times K)\le\sqrt{2}\varepsilon$, and so we may assume that $n>0$. Let $v\in V$. Then using the continuity of f in the first variable, we see that $S(v)\subset S(w)$ if w is sufficiently close to v. Hence there exists an open neighborhood W of v contained in V such that $S(v)\subset S(w)$ for each $w\in W$, but then $S(v)=S(w)$ since $|S(v)|=|S(w)|=n$ for $w\in W$. Write $T=S(v)$.

Let $\varphi\colon J(S)\to J(S)$ be a continuous map given by

$$\varphi(re_s)=\ \max\{r-\varepsilon,0\}e_s\quad \text{for } r\in[0,1] \text{ and } s\in S.$$

Then $\varphi\circ f(W\times K)\subset J(T)$. By our assumption $\mathcal{N}(W,K;\mathbb{R})$ holds and therefore $\mathcal{N}(W,K;\mathbb{R}^\infty)$ holds. Now $J(T)$ is contained in a finite dimensional subspace of the Hilbert space, and hence $\mathcal{N}(W,K;J(T))$ holds. Consequently there exists a $u\in W\subset U$ such that $\varphi\circ f$ is jointly continuous at each point of $\{u\}\times K$. Hence for each $k\in K$, there exists a neighborhood H of (u,k) in $W\times K$ (therefore, in $B\times K$) such that diam $\varphi f(H)\le\varepsilon$. But, for each $x\in J(S)$, $\|x-\varphi(x)\|\le\varepsilon$, and therefore diam $f(H)\le 2\varepsilon+\operatorname{diam}\varphi f(H)\le 3\varepsilon$. ∎

A.2 THEOREM: *Let K be a compact space. If $\mathcal{N}(E,K;\mathbb{R})$ holds for each Tychonoff Baire space E, then $\mathcal{N}(B,K;M)$ holds for each Baire space B and each metrizable space M.*

Proof: Our proof uses a method of [PW; Section 6.6]. By Theorem A.1, it is sufficient to prove that $N(B,K;\mathbb{R})$ holds for each Baire space B or equivalently that, whenever $f\colon B \to (C(K), \tau_p)$ is a continuous map, the set of all points of continuity of $f\colon B \to (C(K), \text{norm})$ is a dense G_δ-set in B.

Let B be a non-empty Baire space, let $\mathcal{B}$ be the Boolean algebra of regularly open sets in B, and let $S(\mathcal{B})$ be the Stone-space of $\mathcal{B}$ (cf. [Ko]). Recall that $S(\mathcal{B})$ is the space of all ultrafilters $\mathcal{F}$ in $\mathcal{B}$ with a compact Hausdorff topology. For each $V \in \mathcal{B}$, let

$$S(V) = \{\mathcal{F} \in S(\mathcal{B})\colon V \in \mathcal{F}\}.$$

Then $\{S(V)\colon V \in \mathcal{B}\}$ is a base for the topology consisting of compact open sets, and $V \mapsto S(V)$ is a Boolean algebra isomorphism of $\mathcal{B}$ and the family of all compact open sets in $S(\mathcal{B})$.

Let

$$E = \{\mathcal{F} \in S(\mathcal{B})\colon \bigcap\{\overline{V}\colon V \in \mathcal{F}\} \neq \emptyset\},$$

and give E the induced topology. Then E is a Tychonoff space. We show that E is a Baire space. Let $\{\Gamma_n\colon n \in \mathbb{N}\}$ be a sequence of dense open subsets of E, and, for each n, let

$$G_n = \bigcup\{V \in \mathcal{B}\colon S(V) \cap E \subset \Gamma_n\}.$$

Then G_n is open and dense in B, and, since B is a Baire space, for each non-empty $U \in \mathcal{B}$, there is an $x \in U \cap \bigcap\{G_n\colon n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a ultrafilter that contains the filter $\{V \in \mathcal{B}\colon x \in V\}$. Then $x \in \bigcap\{\overline{V}\colon V \in \mathcal{F}\}$. (For, otherwise, for some $V \in \mathcal{F}, x \in B\backslash\overline{V}$ and hence $B\backslash\overline{V} \in \mathcal{F}$.) This shows that $\mathcal{F} \in E \cap S(U)$. Furthermore, for each n, there is a $V_n \in \mathcal{B}$ such that $S(V_n) \cap E \subset \Gamma_n$ and $x \in V_n$. Then $\mathcal{F} \in S(V_n) \cap E \subset \Gamma_n$. Consequently $S(U) \cap \bigcap\{\Gamma_n\colon n \in \mathbb{N}\} \neq \emptyset$ for each non-empty $U \in \mathcal{B}$, or equivalently $\bigcap\{\Gamma_n\colon n \in \mathbb{N}\}$ is dense in E.

Since $(C(K), \tau_p)$ is regular Hausdorff, for each $\mathcal{F} \in E, f(\bigcap\{\overline{V}\colon V \in \mathcal{F}\})$ is a singleton, say $\{\varphi(\mathcal{F})\}$. We note that $\varphi(S(V) \cap E) \subset f(\overline{V})$ for $V \in \mathcal{B}$, and hence $\varphi\colon E \to (C(K), \tau_p)$ is continuous using once more the regularity of τ_p. By hypothesis the set of all points of continuity of the map $\varphi\colon E \to (C(K), \text{norm})$ is a dense G_δ-set in E. Now, for $\varepsilon > 0$, let

$$G_\varepsilon = \bigcup\{H\colon H \text{ is open in } B \text{ and } \operatorname{diam} f(H) \leq \varepsilon\}.$$

The proof is complete if the open set G_ε is shown to be dense in B. Suppose not, i.e. $U = B\backslash\overline{G}_\varepsilon \neq \emptyset$. Then $U \in \mathcal{B}$ and there is a point of continuity of

$\varphi: E \to (C(K), \text{norm})$ in $S(U) \cap E$. Hence there exists a non-empty $W \in \mathcal{B}$ such that $W \subset U$ and diam $\varphi(S(W) \cap E) \leq \varepsilon$. As we have seen, for each $x \in W$, there is an $\mathcal{F} \in S(W)$ with $x \in \bigcap\{\overline{V}: V \in \mathcal{F}\}$ and so $\varphi(\mathcal{F}) = f(x)$. It follows that $f(W) \subset \varphi(S(W) \cap E)$ and therefore diam $f(W) \leq \varepsilon$. This contradicts that $W \cap G_\varepsilon = \emptyset$. ∎

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